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Contractor's Report
Publication No. U-2202

THE SPADATS MATHEMATICAL MODEL

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5 August 1963

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Aeronutronic
A Division of Ford Motor Company
Ford Road
Newport Beach, California

Prepared under
Contract AF 19(628)-562

496L Systems Project Office
Electronic Systems Division
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ABSTRACT

The mathematical model presently used by the Space Detection and Tracking System (SPADATS), operated by the U.S. Air Force, Air Defense Command, to describe the orbital motion of Earth satellites is presented. The formulation is given and the non-Keplerian terms are justified. Various output formats of the system are described and defined. Limitations of the model are discussed in a qualitative fashion.

THE SPADATS MATHEMATICAL MODEL

"Glendower: I can call spirits from the vasty deep.

Hotspur: Why, so can I and so can any man;
But will they come when you do call for them?"

Henry IV, Part I, Act 3, Scene 1

Anybody can predict the future positions of Earth satellites. Whether they will be at the predicted place is another matter. In more scientific language: the problem is one of accuracy in prediction. This accuracy or, more realistically, the inaccuracy is due to many causes. The purpose of this paper is to discuss various mathematical models of satellite orbits, the underlying assumption, the errors and limitations of the models. In particular, the SPADATS model will be defined and delimited.

THE NATURE OF MATHEMATICAL MODELS

A mathematical model is an entity in itself. It does not need any physical theory to sustain it. The test of the model is whether it generates predictions of the required accuracy. For instance, by a few days of observation of the rising Sun, it is possible to predict where on the horizon it will rise the next day. After records have been kept for about a year, it should be possible to design a mathematical model for the azimuth of the rising Sun (at a particular site) which includes an annual term. This model would be quite accurate for many years, if we know how long a year is.

This is an important point. The accuracy of the mathematical model depends on the accuracy of its constants. It may take many years of counting the days between the risings of the Sun at a given place on the horizon to refine the value to 365 days and the best part of a century to refine it to 365.24. Why? Because the measurement of azimuth will be inaccurate. It is necessary to define the given place on the horizon accurately and the place to stand. Thus, the accuracy of the constants

depends on the accuracy of the measurements used to obtain the constants.

Such a mathematical model was arrived at, many millenia ago, without the knowledge of any underlying causes. It is possible to build quite an elaborate structure this way.

The Ptolemaic System of Planetary Motions was based on rather unsound principles. Yet, the constants were well determined and the system represented the phenomena quite accurately. The more realistic universe of Aristarchos (fl. 280 B.C.) with the Sun at its center was rejected partly because Aristarchos had to place the stars almost infinitely far away, which seemed ridiculous, and partly because hardly any simplification could be achieved until the Pythagorean assumption of circular motions was overthrown by Kepler.

THE EARTH-SATELLITE PROBLEM

The work of Newton makes it possible to express the problem of the prediction of the future positions of a body in terms of the forces acting on it. One may, therefore, integrate the resultant accelerations numerically or analytically to obtain positions and velocities.

The dominant force on Earth satellites (except during launch or re-entry) is the central attraction of the Earth. If this were the only force, the orbit of the satellite would be a Keplerian ellipse. This type of motion is completely defined by the six elements which are the constants of an analytical integration.

The elements are mathematically equivalent to the position and velocity of the vehicle at any given time. Three elements describe the shape and size of the orbit and the location of the vehicle in the orbit at a specified time The classical dimensional elements are as follows:

a - semi-major axis

e - eccentricity

T - time of perigee passage (i.e., point of closest approach to the Earth)

The remaining three elements specify the orientation of the conic in space. The most descriptive elements are the orientation angles. These angles are as follows:

i - inclination of the orbit with respect to the equatorial plane

Ω - right ascension of the ascending node, the angle in the equatorial plane between the vernal equinox and the line to the ascending node

ω - argument of perigee, the angle in the orbit plane between the line to the ascending node and the line through perigee.

The orientation angles and some related unit vectors are shown in Figure 1. In actual computation, it is preferable to use the components of those vectors (direction cosines) directly and to disregard the angles as much as possible.

There are, however, additional forces acting on the satellite. These may be treated as perturbations on the Kepler solution. In this sense it may be said that the orbit of an Earth satellite is continually changing due to perturbative effects. Though much smaller than the central attraction of the Earth, the perturbative forces can cause deviations from the basic Keplerian ellipse which are appreciable from the standpoint of satellite tracking. Some of the causes that may be important in terms of the SPADATS mission are:

(1) Earth bulge

(2) Atmospheric drag

(3) Solar attraction and radiation pressure

(4) Lunar gravity

The effects of the perturbations are usually grouped into two classes: secular (those which grow with time) and periodic (those which oscillate). The secular class causes the larger errors, because these perturbations build up with time. In some cases and applications, the periodic perturbations of an orbit cannot be ignored.

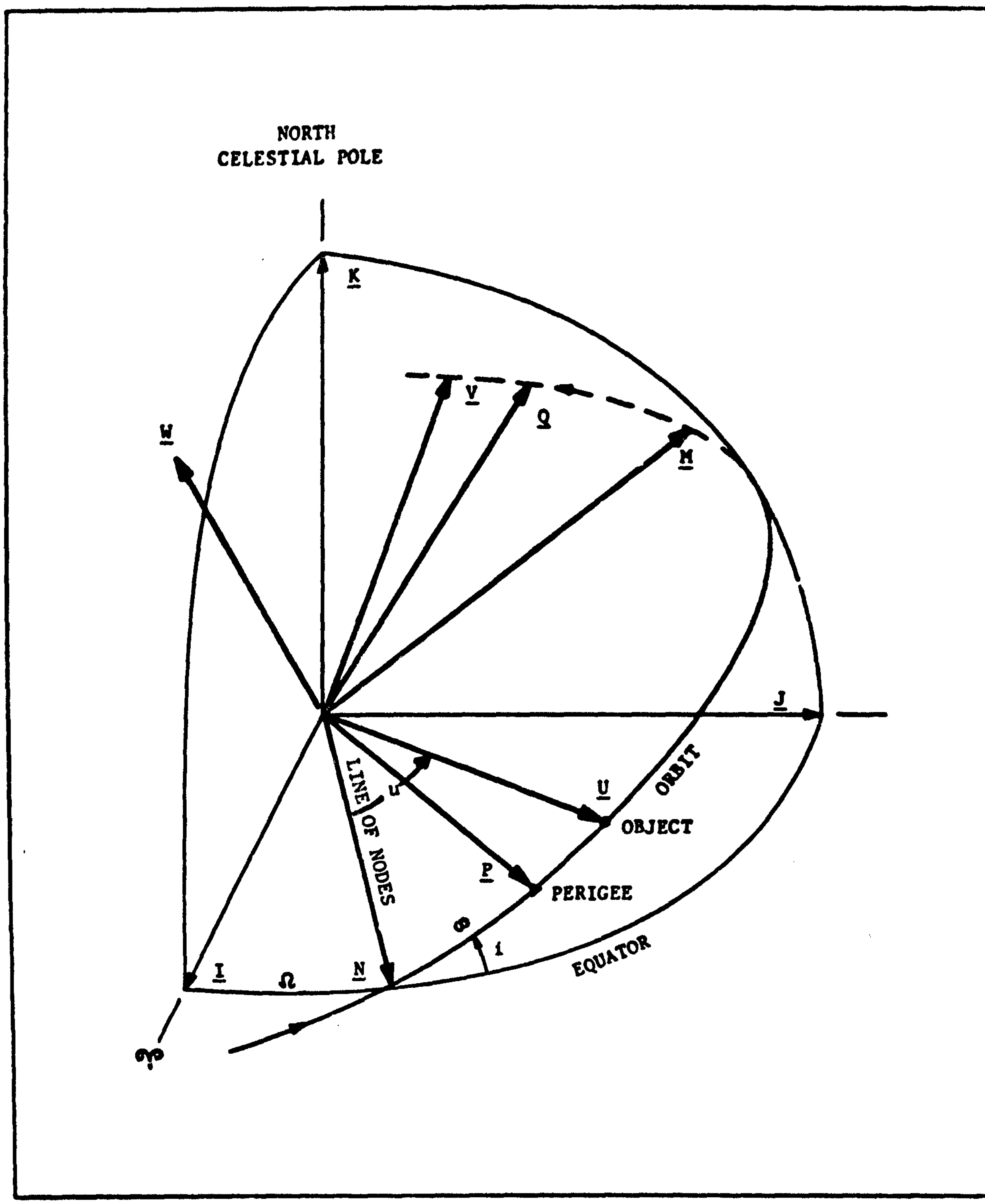

FIG. 2 PROJECTION OF ORBIT ON CELESTIAL SPHERE, WITH ORIENTATION UNIT VECTORS AND ANGLES DISPLAYED

To fulfill the mission of satellite tracking, the future courses of satellites must be predicted with some degree of precision. The perturbative accelerations may be either integrated numerically or by the term-by term analytical integration of a series expansion. The former process is called Special Perturbations in Celestial Mechanics. The analytic integration results in a General Perturbations "theory" of the orbit. One advantage of such a "theory" involves the ability to predict the position for any time without having to compute intermediate positions as is necessary for numerical integration. Again, the gross effects of the perturbative force become more readily apparent from the theory than from inspection of the numbers resulting from Special Perturbations calculations. In the Special Perturbations process the perturbative expressions are generally simpler to express and formulate with comparable, or better, accuracy.

The ultimate in simplicity results when even the conic solution is ignored and the total acceleration is integrated numerically. This process is called "Cowell's Method" to distinguish it from the older methods i.e., Variation of Elements and Encke's Method. Cowell's method became feasible with the development of desk calculators and is especially suitable for large-scale electronic computers; however, it is relatively slow for satellite orbits unless the perturbative acceleration is almost equal to the gravitational acceleration.

General Perturbations theories have been developed to express the effect of the aspherical gravitational field of the Earth, which causes the principal perturbations on a near-Earth satellite during most of its orbital lifetime. The SPADATS mathematical model is based mainly on the work of Kozai and Brouwer (See Appendix C).

It is possible to make an increasingly accurate integration of the equations of motion in specific cases; however, it is usually unwarranted to complicate the solution because the apparent satellite path will deviate from the true path because of inexact observations, inexact theory and constants (model errors), and inexact computations (e.g., round-off). The majority of the Earth satellite prediction needs can be met practically by a relatively simple General Perturbations treatment in conjunction with a correction procedure which employs observational data. This correction is an integral part of any computation of an actual orbit because it establishes the elements of the orbit. The form and size of the error in the elements depends not only on the observations used in the correction but somewhat on the correction process also.

THE DIFFERENTIAL CORRECTION

In order to correct an orbit, it is necessary to "represent" the observations (i.e., calculate what the observations would have been had the elements and the mathematical model for the orbit representation been correct). The end products of the observation processing are the residuals which are the differences: the observed quantities minus the "represented" quantities.

The calculation of the represented observation may include all components of a complete "fix" (three components of position and three components of velocity). However, residuals can be obtained for only those components which are actually observed. Some of these residuals may not be usable if the observing instrument performs poorly in some aspects of measuring. Nevertheless, the remaining residuals may form the basis for differentially correcting an orbit.

When there is redundancy in the observations, i.e., when there are more residuals than there are parameters to be corrected (normally seven), the differential correction process becomes even more effective, since the process makes use of the method of statistical averaging known as least squares. Observations which are separated by long intervals of time and which are obtained from different points on the Earth can be included.

The observations are weighted, in the present system, according to certain geometric effects. These weighting factors are given in Table I. Except for the range-rate residual, these factors express the physical displacement (in kilometers) of the vehicle from its assumed place as indicated by the observed quantity. This physical displacement component is also used to reject those equations of condition which indicate a displacement greater than an absolute distance criterion or greater than 1.5 times the root-means-square (r.m.s.) of the displacement component.* The equations for range-rate are tested against a separate but similar absolute criterion and against 1.5 times the r.m.s. of the remaining range-rate residuals. This rejection process may be considered a 0 or 1 weighting on the basis of the internal consistency of the data. The factors cos h and cos δ account for the degradation of azimuth and right ascension measures near the zenith and the poles, respectively. Weighting on the basis of sensor characteristics is not performed at present.

*remaining after rejection or the basis of the absolute criterion.

TABLE I

MULTIPLICATIVE FACTORS OF EQUATIONS OF CONDITION

Observed Quantity	Factors	Residual Form
Range	None	$\Delta\rho$
Range-Rate	Range	$\rho\Delta\dot{\rho}$
Azimuth	Range, Cosine Elevation	$\rho\cos h \ \Delta A$
Elevation	Range	$\rho\Delta h$
Right Ascension	Range, Cosine Declination	$\rho\cos\delta\Delta\alpha$
Declination	Range	$\rho\Delta\delta$

THE SPADAT SYSTEM ORBIT REPRESENTATION

The angles which are used in the classical element set have the disadvantages of singularities and computational ineffiencies. For example, when the eccentricity becomes small, the argument and time of perigee become ill-defined. When the inclination is small (nearly 0° or nearly 180°), the node is ill-defined. As usual, the exact singularities have orbit representations which are often simpler than those for the general conic. However, the near-singularities may cuase difficulties in both representation and correction.

Some of these difficulties can be avoided, and computational efficiency can be gained, by use of an element set based on vectors (direction cosines) instead of angles. Such a set is the Element Set listed below. The relationships between the orientation vectors and the orientation angles are illustrated in Figure 1.

TABLE II

INTERNAL ELEMENTS OF SPADAT SYSTEM

t_o = epoch time; can be selected at an arbitrary position in the orbit

a_{xN}, a_{yN} = components of $\underline{a}$ along $\underline{N}$ and $\underline{M}$, respectively; where $\underline{a} = e\underline{P}$, $a_{xN} = e \cos \omega$, $a_{yN} = e \sin \omega$

$\underline{P}$ is a unit vector directed toward perigee from the center of the Earth

$\underline{h}$ $= \sqrt{p}\underline{W}$, angular momentum vector, where $\underline{W}$ is the normal to the orbit plane (See Figure 1) and p is the semi-latus rectum

$$L_o = \omega_o + M_o + \frac{\cos i}{|\cos i|} \Omega_o$$

= mean longitude at epoch

where M_o, ω_o, Ω_o, are the mean anomaly, argument of perigee, and right ascension of the node, all at epoch

$$\left.\begin{array}{ll} c & = \text{period decay rate} \\ d & = \text{period decay acceleration} \end{array}\right\} \text{drag parameters}$$

The formulation presently used to obtain position and velocity from these elements is contained in Appendix A. The model includes the first-order secular terms in L, ω and Ω, due to the second zonal harmonic of the geopotential, as well as the large long-period variations in L and a_{yN} due to the third zonal harmonic. The derivation of the formulas involving those harmonics is given in Appendix C. The definition of output elements in terms of those given in Table II is discussed in Appendix B.

LIMITATIONS

The principal limitation of the SPADATS Model is in accuracy.* To preserve computational efficiency only a limited number of perturbative terms are included in the representation. No short-period effects are included. The elements determined by the differential correction are mean elements at the epoch, t_o, that is, they are intended to be the elements which would represent the orbit if all the periodic terms were zero at t_o.

The elements are produced by the SPADAT center in various forms. Several of these forms are defined in Appendix B. One form, the Space Track Bulletin, is no longer widely circulated. It consists of four parts. Part I contains some of the same information as on the 7-Element Cards. Part II gives the times, longitudes and heights of a set of ascending nodal crossings. Part III gives the time and longitude corrections to convert the information from Part II to other latitude crossings. Part III is calculated for some median revolution of Part II. Part IV is the SATOR code, giving the elements in a telegraphic format. Since it receives wider distribution, the SATOR code is also defined in Appendix B.

It must be borne in mind that the elements (and the bulletins) are issued for purposes of prediction. The elements are determined from observations made prior to their epoch. The bulletin elements, for example, are best for that preceding period and not for the revolutions indicated in PartII. Furthermore, they are the mean elements from a truncated model and cannot be used to define starting conditions for a numerical integration of the orbit.

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*Quantitative information on the accuracy of the present SPADAT System may be found in Aeronutronic Report S-2071, "Spadat System Observational Data Studies," Volume I of <u>Studies of Satellite Prediction Accuracy</u>, ESD-TDR-63-123, by G. R. Miczaika and D. M. Slager.

It may be thought that it is possible to obtain osculating elements by adding the periodic terms to the mean elements. Such osculating elements may also be inaccurate. The differential correction obtains the best set of mean elements from the observational material presented to it. The associated least-squares process is designed to minimize the transfer of random errors to the elements. It is limited, however, by the absence of some of the periodic terms in the representation. Least-squares would be able to smooth out these small variations if the observations presented it with a sample of these effects that is random with respect to phase. The sample of the short-periodic effects is biased by the preponderance of sensors in the Northern Hemisphere and by a frequent concentration of observations at the northern apex of the orbits. The sample of the long-period effects is biased because of the limited time-span of observations used in a typical differential correction. This time-span, dictated usually by the irregular variations of atmospheric drag, may not allow a significant arc of long-period terms to be included.

Assume that the observations are so concentrated, for one of the foregoing reasons, that they sample some neglected periodic term in a small part of a full cycle (See Figure 2). Then the SPADATS "mean" element or position would tend to include the mean value of the neglected term during the observation span. The figure indicates that the epoch is taken at the end of the observation span. This would be the case if the neglected term had a long period, since the SPADAT System places the epoch at a nodal crossing near the last observation. It can be seen that adding the value of the neglected term to the SPADATS mean does not necessarily give the osculating value at the epoch and may, indeed, give a more inexact value. This indirect effect of the neglected terms is somewhat mitigated by including in the differential correction different types of observations (e.g., range-rate as well as range) which simultaneously sample the same periodic effects at different phases.

The deviation of the SPADATS Model from the true orbit is kept in bounds by testing the new observations (after epoch). If the root-mean-square of the displacement of the observed positions from the positions computed from the model exceeds a bound, a new correction is called for by the system.

Another type of limitation concerns the applicability of the semi-automatic programming system, which uses the appended formulation. This system is only designed to handle near-Earth satellites. Separate programs must be used to predict the position of lunar probes, space probes, and equatorial satellites.

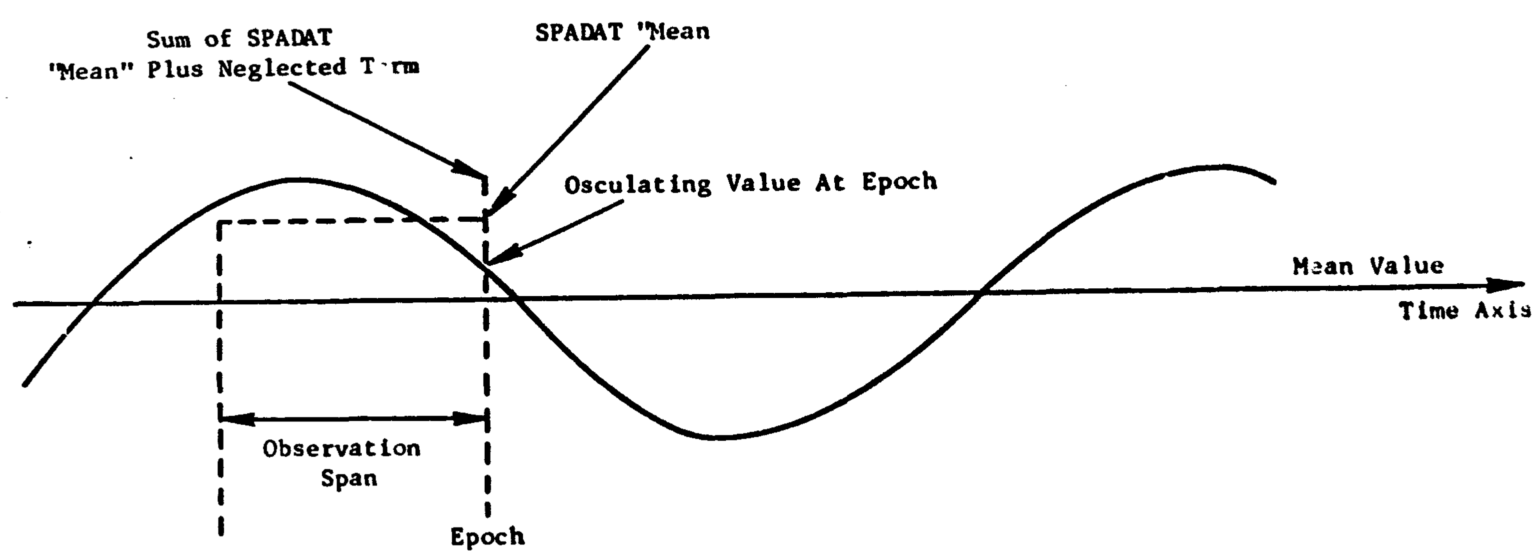


FIGURE 2. EFFECT OF NEGLECTED PERIODIC TERM

SATØR CODE

The SATOR Code is produced as Part IV of the Bulletin by the BLTNSGP program. The format is as follows:

SATØR	aabbc	deeff	ggggZ	hhhhX	NØWES
iiiii	jkkkk	ARPER	11111	mnnnX	PERIØD
ooooo	ppppp	ECCEN	qqqqq	PERRA	rrrrr
RAFRE	sssss	(sssss repeated as necessary)		RTASC	ttttt

Key:

aa = Last two digits of year satellite launched
bb = Greek letter designation, 01 - Alpha, 02 - Beta, etc., or number after 1963
c = Component A=1, B=2, etc.
d = Reference time (time of perigee passage closest to epoch): last digit of numerical notation for month; i.e., 1=January or November, 2=February or December, 3=March, etc.
ee = Reference time: date
ff = Reference time: hour
gggg = Reference time: minutes and hundredths of minutes
Z = Universal time, Greenwich Mean Time
hhhh = Inclination in degrees and hundredths of degrees. If the orbit inclination is negative (satellite fired westward) group is preceded by NEGAT.
X = Always an X
NØWES = Subindicator for geographical longitude of northbound node west of Greenwich at reference time
iiiii = West longitude of northbound node, λ_w, in degrees and hundredths of degrees,
j = 1. If plus: When the "prime sweep interval" is one day plus a certain number of minutes.
2. If minus: When the "prime sweep interval" is one day minus a certain number of minutes. This is equivalent to saying that the same portion of the orbit plane will reappear at the same location a certain number of minutes earlier each day.

kkkk = Number of minutes by which "prime sweep interval" differs from one day or 1440 minutes. This is another way of expressing the relative motion of the orbit plane.

ARPER = Subindicator (argument of perigee) angular distance of perigee from node at reference time. For modified orbital elements, this is also the position of the satellite in the ellipse at reference time (mean anomaly at epoch is always equal to zero in this system)

11111 = ω, angular distance of perigee from northbound node, measured in the direction of satellite travel in degrees and hundredths of degrees

m = 1 for plus, if perigee moves in the same direction as satellite travel. 2 for minus, if perigee moves in the direction opposite to satellite travel

nnn = $P_a\ d\omega/dt$, average decimal fraction of a degree which perigee moves per period, measured in thousandths of a degree

X = Always an X

PERIØD = Subindicator for perigee-to-perigee period (anomalistic period)

ooooo = Anomalistic period, P_a, in minutes and thousandths of a minute. If first two digits are less than 85, it should be understood that 100 should be added in order to arrive at the correct period. Should the period be greater than 185 minutes, a special notation will be made in the message.

ppppp = $P_a\ dP_a/dt = -2c''P_a^2$ change in anomalistic period per revolution, measured as a decimal fraction in one hundred thousandths of a minute

ECCEN = Subindicator for eccentricity

ggggg = Eccentricity, measured as a decimal fraction in one hundred thousandths

PERRA = Subindicator for radial distance of satellite from center of earth to perigee

rrrrr = q, radial distance of satellite from center of earth at perigee, measured in miles and tenths of miles

RAFRE = Subindicator for radio frequencies currently being transmitted from satellite

sssss = Radio frequency in megacycles and hundredths of megacycles

RTASC = Subindicator for right ascension of the node in order that this message may also serve the needs of those who prefer traditional orbital elements

tttt = Ω , Degrees and hundredths of degrees

Since all of the above quantities have subindicators, a message need only include those quantities which have changed since the last reference time.

SATØR code output is computed from the internal elements for the reference time, the time of perigee passage, T, closest to epoch:

$$M_o = L_o - \omega_o - \Omega_o \text{ if } \cos i > 0$$

$$M_o = L_o - \omega_o + \Omega_o \text{ if } \cos i < 0$$

$$T - t_o = - M_o / n_o$$

Compute west longitude of the ascending node λ_{W_Ω} (T), and argument of perigee, ω(T), at T:

θ_{gr} (T) (deg) = θ_{gr}(at start of epoch year) + $\dot{\theta}$ - 360°) D + $\dot{\theta}$ f where $\dot{\theta}$ is the rotation rate of the earth in deg/ solar day, D and f are T in days and fractions of a day respectively from start of epoch year.

$$\Omega(T) = \Omega_o + \frac{d\Omega}{dt} (T - t_o)$$

$$\lambda_{W_\Omega}(T) = \theta_{gr}(T) - \Omega(T)$$

$$\omega(T) = \omega_o + \frac{d\omega}{dt}(T - t_o)$$

The "prime sweep interval" is the mean solar time interval between the passage of the orbit plane over a geographic location. Specificially, it is desirable to consider the recurrences of the passage of the ascending node over the Greenwich meridian. Then it is seen that the difference of the mean solar time of this event from one day to the next, Δt, is due to two causes: the revolution of the Earth about the Sun and the regression of the nodes. In minutes this difference is

$$\Delta t = \left\{ \left(\frac{360^\circ}{\dot{\theta} - \frac{d\Omega}{dt}} \right) - 1 \right\} 1440$$

where $\dot{\theta}$ and $\frac{d\Omega}{dt}$ are in degrees per day.

APPENDIX A

SPADATS ORBIT REPRESENTATION

The following formulation is shortly to be used by SPADATS to compute ephemerides. It has undergone minor changes since the latest documentation (SPADATS Programming Document, Aeronutronic Publication U-1691, Revised 1 October 1962), and will probably change further in the future.

The ephemeris calculation in SPADATS computes a mean geocentric position $\underline{r}$ and velocity $\dot{\underline{r}}$ at some time t from the epoch orbit elements a_{xN_o}, a_{yN_o}, $\underline{h}_o$, and L_o, and the drag parameter c_o, at some epoch time t_o. The perturbations caused by the Earth's atmosphere and by the asphericity of the Earth (terms due to the second and third zonal harmonics) are included in this calculation.

The constants used in this model include mean equatorial radius of the Earth:

$$a_e = 6378165 \text{ meters}$$

Second Zonal Harmonic Coefficient

$$J_2 = 1.08245 \times 10^{-3}$$

Third Zonal Harmonic Coefficient

$$J_3 = -2.50 \times 10^{-6}$$

A.1 PRELIMINARY COMPUTATIONS

The following quantities, required in later calculations, are first determined.

(a) Compute the semi-latus rectum, p_o, the eccentricity, e_o, the mean semi-major axis, a_o, the perigee distance q_o, the mean value of the mean motion n_o:

$$p_o = \underline{h}_o \cdot \underline{h}_o = h_{x_o}^2 + h_{y_o}^2 + h_{z_o}^2$$

$$e_o = \left(a_{xN_o}^2 + a_{yN_o}^2\right)^{\frac{1}{2}}$$

$$a_o = \frac{p_o}{1 - e_o^2}$$

$$q_o = a_o (1 - e_o)$$

(A-1) $$n_o = \frac{k_e \sqrt{\mu}}{a_o^{3/2}} \left[1 - \frac{3}{4} J_2 \left(\frac{a_e}{p_o}\right)^2 \sqrt{1 - e_o^2} \, (1 - \frac{3}{2} \sin^2 i_o)\right]$$

(b) Compute the orientation angles i_o and Ω_o:

$$i_o = \cos^{-1} \frac{h_{z_o}}{\sqrt{p_o}} \; ; \; 0 \leq i_o \, \pi$$ (used for output only)

$$\Omega_o = \tan^{-1} \frac{h_{x_o}}{-h_{y_o}}$$; the quandrant is determined from the signs of the numerator (sine) and denominator (cosine).

(c) Compute the mean argument of latitude, U_o:

If $h_{z_o} \geq 0$ (direct motion),

$$U_o = L_o - \Omega_o \quad .$$

If $h_{z_o} < 0$ (retrograde motion),

$$U_o = L_o + \Omega_o \quad .$$

(d) Compute the drag coefficients c" and d:

(A-2) $$c'' = \frac{-360\, n_o^2\, c_o}{\pi^2}$$

(A-3) $$d = A\, (c'')^2 \left[1 + \frac{n_D}{3(n_D - n_o)} \right]$$

$n_D = 0.072\ 220\ 521$

The empirical coefficient A = 8 (at present)

A.2 ATMOSPHERIC DRAG AND ASPHERICITY OF THE EARTH

(a) Compute the perturbations caused by the Earth's atmosphere on a, e, and p during the time interval $(t - t_o)$. At time t:

(A-4) $$a = a_o \left[1 + 2\, c''\, (t - t_o) + 3\, d\, (t - t_o)^2 \right]^{\frac{2}{3}}$$

$$e = 1 - \frac{q_o}{a} \text{ for } a \geq q_o$$

$$e = 0 \text{ for } a < q_o$$

$$p = a(1 - e^2)$$

(b) Using the initial values p_o, n_o, and i_o, compute the secular effects of the Earth's second zonal harmonic, J_2, on the elements Ω and ω :

$$\text{(A-5)} \qquad \frac{d\Omega}{dt} = k_e\dot{\Omega} = -\frac{3}{2} J_2 \frac{a_e^2}{p_o^2} n_o \cos i_o$$

$$\text{(A-6)} \qquad \frac{d\omega}{dt} = k_e\dot{\omega} = \frac{3}{4} J_2 \frac{a_e^2}{p_o^2} n_o (4 - 5 \sin^2 i_o).$$

(c) Compute Ω at time t and the change in ω :

$$\Omega = \Omega_o + \frac{d\Omega}{dt}(t - t_o)$$

$$\omega_s \triangleq \omega - \omega_o = \frac{d\omega}{dt}(t - t_o)$$

where $(t - t_o)$ is the time interval in minutes

(d) Compute a_{xN} and a_{yN} at time t

$$a_{xN} = \frac{e}{e_o} \left[a_{xN_o} \cos \omega_s - a_{yN_o} \sin \omega_s \right]$$

$$a_{yN} = \frac{e}{e_o} \left[a_{xN_o} \sin \omega_s + a_{yN_o} \cos \omega_s \right] - \tfrac{1}{2} \frac{J_3}{J_2} \frac{a_e}{p_o} \sin i_o$$

The $\frac{J_3}{J_2}$ term in a_{yN} represents the leading long-period perturbation caused by the Earth's third zonal harmonic.

(e) Compute the mean longitude at time t, as perturbed by the Earth's second zonal harmonic ($\Delta\pi$ term), the Earth's third zonal harmonic (L_3 term), and atmospheric drag (c and d terms):

(A-8) $$L = L_o + n_o (1+ \Delta\pi) (t-t_o) + L_3 + n_o c (t-t_o)^2 + n_o d (t-t_o)^3$$

where

(A-9) $$\Delta\pi = \frac{3}{2} J_2 \left(\frac{a_e}{p_o}\right)^2 \left(2 - \frac{5}{2} \sin^2 i_o - | \cos i_o | \right)$$

(A-10) $$L_3 = - \frac{1}{4} \frac{J_3}{J_2} \left(\frac{a_e}{p_o}\right) a_{xN} \sin i_o \left[\frac{3 + 5 | \cos i_o |}{1 + | \cos i_o |} \right]$$

A.3 CALCULATION OF GEOCENTRIC POSITION AND VELOCITY VECTORS

(a) Calculate the mean argument of latitude U at time t:

$U = L - \Omega$ if $i \leq 90^o$ (direct motion)

$U = L + \Omega$ if $i > 90^o$ (retrograde motion)

(b) Solve the following modified form of Kepler's equation for the quantity $(E + \omega)$ by iteration, using a first guess of U (mod 2π)

$$(E + \omega) = U + a_{xN} \sin (E + \omega) - a_{yN} \cos (E + \omega)$$

(c) Compute the mean geocentric position $\underline{r}$ and velocity $\dot{\underline{r}}$ at time t by means of the following sequence of equations:

$$e \cos E = a_{xN} \cos (E + \omega) + a_{yN} \sin (E + \omega)$$

$$e \sin E = a_{xN} \sin (E + \omega) - a_{yN} \cos (E + \omega)$$

$$r = a (1 - e \cos E)$$

$$\dot{r} = \frac{\sqrt{\mu a}}{r} e \sin E$$

$$r\dot{v} = \frac{\sqrt{\mu a}}{r} \sqrt{1 - e^2}$$

$$\cos u = \frac{a}{r} \left[\cos (E + \omega) - a_{xN} + a_{yN} \frac{e \sin E}{1 + \sqrt{1 - e^2}} \right]$$

$$\sin u = \frac{a}{r} \left[\sin (E + \omega) - a_{yN} - a_{xN} \frac{e \sin E}{1 + \sqrt{1 - e^2}} \right]$$

$$\underline{U} = \underline{N} \cos u + \underline{M} \sin u$$

$$\underline{V} = - \underline{N} \sin u + \underline{M} \cos u$$

$$\underline{r} = r\underline{U}$$

$$\dot{\underline{r}} = \dot{r}\underline{U} + r\dot{v} \underline{V}$$

APPENDIX B

DEFINITION OF OUTPUT ELEMENTS

This appendix defines the relationships between the internal elements used by the SPADATS Model and the various output formats. It also presents some of the formulation used to update elements after a differential correction.

UPDATE OF ELEMENTS

The quantities corrected are the mean argument of latitude, U_o, the drag coefficient, c", the mean value of the mean motion at the epoch, n_o, and the parameters, a_{xN}, a_{yN}, Ω and i, defined in the text. From the new, corrected values of these quantities are derived the period decay rate,

$$\text{(B-1)} \qquad c_o = \frac{-\pi^2}{360} \frac{c''}{n_o^2} \qquad \text{(days/rev}^2\text{)}$$

and the mean semi-major axis at the epoch,

$$\text{(B-2)} \qquad a_o = \left(\frac{k_e\sqrt{\mu}}{n_o}\right)^{2/3} \left[1 - \frac{3}{4} \frac{J_2 a_e^2}{p_o^2} \left(1 - \frac{3}{2} \sin^2 i_o\right) \sqrt{1-e_o^2} \right]^{2/3} .$$

These formulas are exact inverses of equations (A-1) and (A-2). Those elements experiencing secular variations must be updated to the new epoch, t_o, which is taken at the crossing of ascending node nearest the latest observation. As in Appendix A, the rates include

$$\text{(A-3)} \qquad d = A(c'')^2 \left[\frac{4n_D - 3n_o}{3n_D - 3n_o} \right]$$

where n_D = 0.072220521. This formula is analogous to one found by Findley. The constant A=8 was determined by experimentation*.

$$\text{(A-5)} \qquad \frac{d\Omega}{dt} = -\frac{3}{2} \frac{J_2 a_e^2}{p_o^2} n_o \cos i$$

$$\text{(A-6)} \qquad \frac{d\omega}{dt} = \frac{3}{4} \frac{J_2 a_e^2}{p_o^2} \left[4 - 5 \sin^2 i \right] n_o$$

$$\text{(A-9)} \qquad \Delta\pi = \left[\frac{3}{2} \frac{J_2 a_e^2}{p_o^2} \left(2 - \frac{5}{2} \sin^2 i - | \cos i | \right) \right]$$

The most complicated update formula is that for L

$$L_p = L_o + n_o (1 + \Delta\pi) (t_p - t_o) + n_o c'' (t_p - t_o)^2 + n_o d(t_p - t_o)^3$$

It can be seen that the periodic term, L_3, which appeared in Eq. A-8 is absent because this is a transformation from mean elements to mean elements at a new epoch.

The argument of perigee and right ascension of the node must be updated

$$\omega_o = \tan^{-1} \left(\frac{a_{yN_o}}{a_{xN_o}} \right)$$

* For details see Aeronutronic Monthly Status Report to the 496L SPO MPR-62-181, October 1962, p. 9.

$$\omega_p = \omega_o + \frac{d\omega}{dt} (t_p - t_o)$$

$$\Omega_p = \Omega_o + \frac{d\Omega}{dt} (t_p - t_o)$$

The drag parameters cause a change in the semi-major axis

(B-3) $$a_p = a_o \left[1 + 2c'' (t_p - t_o) + 3d (t_p - t_o)^2 \right]^{-2/3}$$

During the calculation of the new elements, the value of the drag coefficient becomes

(B-4) $$c''_p = c'' + 3d (t_p - t_o).$$

FOUR-LINE ORBITAL ELEMENT FORMAT FOR NORAD SPADATS USE

The orbital element standard format for NORAD SPADATS use has been established to allow Teletype transmission and automatic tape-to-card or card-to-tape conversion. Line 0 is used only for descriptive material of primary interest in connection with the issuance of bulletins. Lines 1 through 3 contain all the numerical values of the orbital elements used in computations and some additional values of interest (the anomalistic period, the perigee height, and the rate of change of the period) which would not be used in calculations. Finally a check sum is placed at the end of each of lines 1 through 3 to allow detection of transmission errors.

A sample message as it would appear in print is as follows:

```
0   005   312   58 BETA       002 US11 62 04 27 314 04 29  98
1   005   312   37781.00000000 01.35993597 -.168030-06 -.110000-10  120
2   005   312   034.2490 275.4970 297.2870 .1862006 0133.99 000681  167
3   005   312   156.2020 004.4050 -03.0140 -.612-04 -.140-3 016243  089
```

LINE 0 SATELLITE AND BULLETIN DESCRIPTION

Col. 1	Line Number (0)		
Col. 2	Space		
Cols. 3-6	Satellite Number		
	Col. 3	Space if SPADATS Number (Digit 4 through 9 if SPASUR Number used prior to issue of SPADATS Number)	
	Col. 4-6	Satellite Number	
Col. 7	Space		
Col. 8	Source of Elements		
		Space if SPADATS 9 if SPASUR	
Cols. 9-11	Element Set Number		
Col. 12	Space		
Cols.13-28	Astronomical Name		
	Cols. 13-14	Last two digits of year	
	Col. 15	Space	
	Cols. 16-24	Greek letter (before 1963) or Numeral	
	Col. 25	Space	
	Cols. 26-28	Piece number (before 1963) or Letters	
Col. 29	Space		
Cols.30-33	Country of Origin and Type		
	Cols. 30-31	Code	Country
		US	United States
		SR	Russia
		FR	France
		UK	England
		CA	Canada
		JA	Japan
		Others	As assigned
	Col. 32	Code	Type
		0	Unknown
		1	Scientific
		2	Weather
		3	Navigation
		4	Communications
		5	Manned
		6-9	As assigned

LINE 1 SATELLITE ORBITAL INFORMATION

Col. 1	Line Number (1)		
Cols. 2-12	Same as Line 0		
Cols. 13-26	Epoch, T, Modified Julian Days (Julian Day minus 2,400,000.5)		
		Cols. 13-17	Integral part
		Col. 18	Decimal point
		Cols. 19-26	Decimal fraction
Col. 27	Space		
Cols. 28-38	Semi-major Axis, a, Earth equatorial radii		
		Cols. 28-29	Integral part
		Col. 30	Decimal point
		Cols. 31-38	Decimal fraction
Col. 39	Space		
Cols. 40-50	First Time Derivative of Semi-major axis, da/dt, Earth radii/day		
		Col. 40	Sign
		Col. 41	Decimal point
		Cols. 42-47	Decimal fraction
		Col. 48	Sign of exponent
		Cols. 49-50	Exponent of 10

(B-5)

$$\frac{da}{dt} = - \frac{4}{3} a_o c''$$

Col. 51	Space		
Cols. 52-62	Second Time Derivative of Semi-major axis d^2a/dt^2, Earth radii/day/day		
		Col. 52	Sign
		Col. 53	Decimal point
		Cols. 54-59	Decimal fraction
		Col. 60	Sign of exponent
		Cols. 61-62	Exponent of 10

(B-6)

$$\frac{d^2a}{dt^2} = - 4a_o d + \frac{40}{9} a_o (c'')^2$$

Col. 63	Space
Cols. 64-66	Check Sum (Arithmetic sum of digits in line, plus one for each minus sign)

Col. 33		Code	Type
		0	Silent
		1	Transmitting
		2	Rocket body-silent
		3	Rocket body-transmitting
		4	Metal or other fragment
		5-8	As assigned
		9	Unspecified

Col. 34	Space	
Cols.35-42	Epoch	
	Cols. 35-36	Last two digits of year
	Col. 37	Space
	Cols. 38-39	Month
	Col. 40	Space
	Cols. 41-42	Day
Col. 43	Space	
Cols.44-46	Bulletin Number	
Col. 47	Space	
Cols.48-52	Date of Bulletin Issue	
	Cols. 48-49	Month
	Col. 50	Space
	Cols. 51-52	Day
Col. 53	Space	
Cols.54-56	Length of Bulletin	Number of revolutions

LINE 2 SATELLITE ORBITAL INFORMATION

Col. 1	Line Number (2)	
Cols. 2-12	Same as Card 0	
Cols. 13-20	Inclination, i, Degrees	
	Cols. 13-15	Integral part
	Col. 16	Decimal point
	Cols. 17-20	Decimal fraction
Col. 21	Space	
Cols. 22-29	Argument of Perigee, ω, Degrees	
	Cols. 22-24	Integral part
	Col. 25	Decimal point
	Cols. 26-29	Decimal fraction
Col. 30	Space	
Cols. 31-38	Right Ascension of the Ascending Node, Ω, Degrees	
	Cols. 31-33	Integral part
	Col. 34	Decimal point
	Cols. 35-38	Decimal fraction
Col. 39	Space	
Cols. 40-47	Eccentricity, e, (dimensionless)	
	Col. 40	Decimal point
	Cols. 41-47	Decimal fraction
Col. 48	Space	
Cols. 49-55	Anomalistic Period, P_a, Minutes	
	Cols. 49-52	Integral part
	Col. 53	Decimal point
	Cols. 54-55	Decimal fraction

$$P_a = \frac{2\pi}{n_o} \qquad \text{(B-7)}$$

Col. 56	Space
Cols. 57-62	Height of Perigee Above Earth's Equatorial Radius, H_p, Kilometers

$$H_p = (a-ae-1)\ a_e$$

Col. 63	Space
Cols. 64-66	Check Sum (Described in Line 1)

LINE 3 SATELLITE ORBITAL INFORMATION

Col. 1	Line Number (3)		
Cols. 2-12	Same as Line 0		
Cols. 13-20	Mean Longitude, L_o, Degrees		
		Cols. 13-15	Integral part
		Col. 16	Decimal point
		Cols. 17-20	Decimal fraction
Cols. 21	Space		
Cols. 22-29	First Time Derivative of Argument of Perigee, $d\omega/dt$, Degrees per day		
		Col. 22	Sign
		Cols. 23-24	Integral part
		Col. 25	Decimal point
		Cols. 26-29	Decimal fraction
Col. 30	Space		
Cols. 31-38	First Time Derivative of Right Ascension of Ascending Node, $d\Omega/dt$, Degrees per day		
		Col. 31	Sign
		Cols. 32-33	Integral part
		Col. 34	Decimal point
		Cols. 35-38	Decimal fraction
Col. 39	Space		
Cols. 40-47	First Time Derivative of Eccentricity, de/dt, per day,		
		Col. 40	Sign
		Col. 41	Decimal point
		Cols. 42-44	Decimal fraction
		Col. 45	Sign of exponent
		Cols. 46-47	Exponent of 10

(B-8) $$\frac{de}{dt} = -\frac{4}{3}(1-e)\,c''$$

Col. 48	Space		
Cols. 49-55	First Time Derivative of Anomalistic Period, dP_a/dt, Minutes per day		
		Col. 49	Sign
		Col. 50	Decimal point
		Cols.51-55	Decimal fraction in floating point notation

(B-9) $$\frac{dP_a}{dt} = -2c'' P_a$$

Col. 56	Space
Cols. 57-62	N_o, Epoch Revolution*
Col. 63	Space
Cols. 64-66	Check Sum (Described in Line 1)

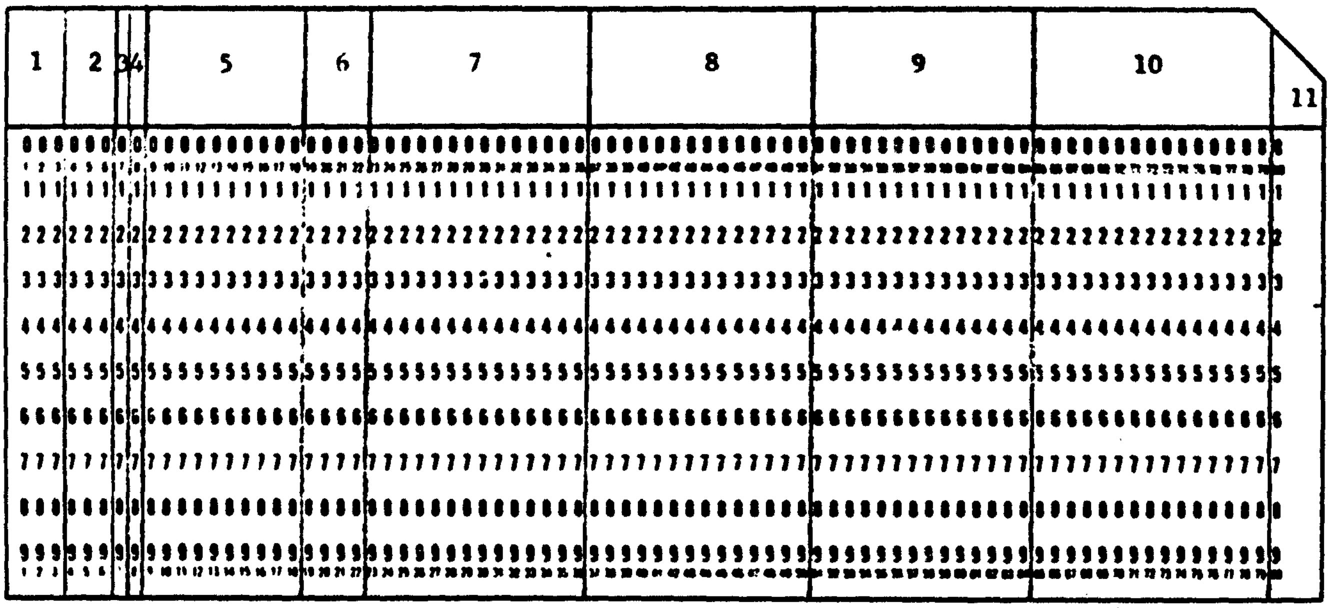
SEVEN-CARD ELEMENT FORMAT

The elements used in the formulation described in Appendix A are introduced into the computer (and produced by the computer) in the form of seven cards shown in Figures 3-9. Card 5 is virtually not used by the system. Card 7 contains bookkeeping information concerning the beginning and expiration dates of bulletins and look angles. The nodal period, P_N, on card 6 is calculated by

(B-10) $$P_N = \frac{2\pi}{n_o} \left[1 - \frac{3}{4} J_2 \left(\frac{a_e}{P_o} \right)^2 (4 - 5 \sin^2 i) \right]$$

* Launch is assumed to occur in Revolution 0. Revolution 1 starts at the first crossing of the ascending node. The revolution count is not always accurate because in some cases revolutions have been lost due to lack of sufficient observations. If the exact time of launch is unknown, an arbitrary Revolution 1 may be defined.

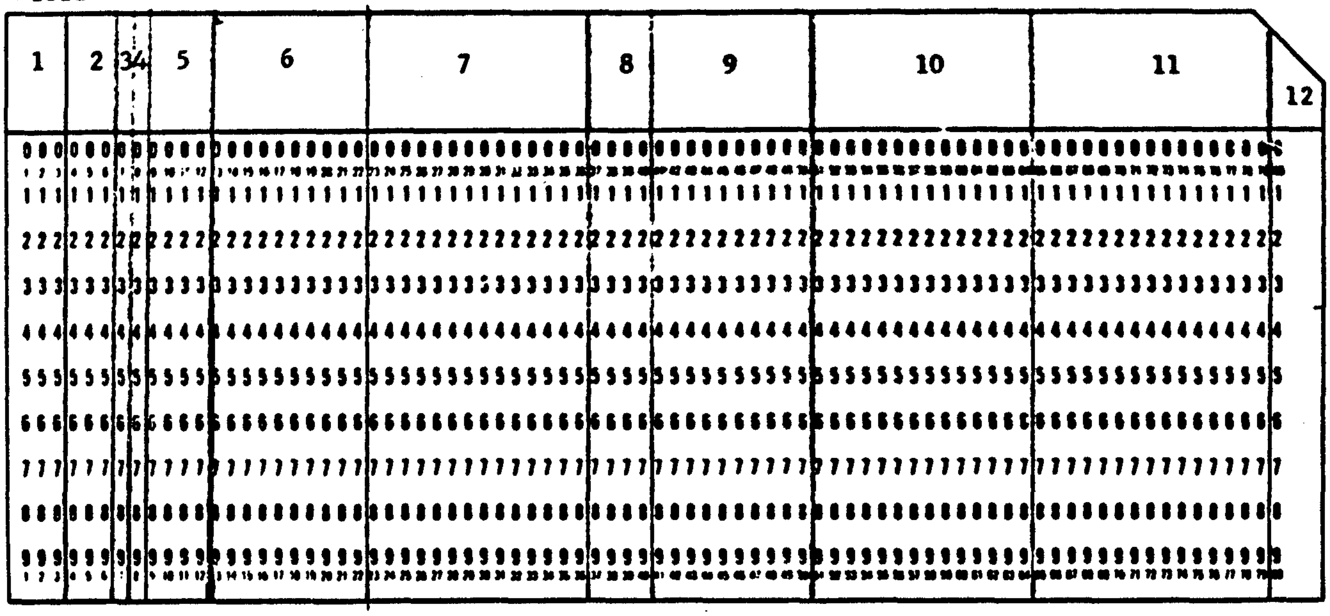
Field



Field No.	Columns	Description
1	1 - 3	Satellite number - justified right
2	4 - 6	Element set number - justified right
3	7	Not used
4	8	Card number (Card # = 1)
5	9 - 18	Satellite name for Element File Update
6	19 - 22	Not used
7	23 - 36	N_o - Epoch revolution
8	37 - 50	e - Eccentricity
9	51 - 64	i - Inclination (degrees)
10	65 - 79	Not used
11	80	Card type E = Nodal Elements

FIGURE 3 ELEMENT CARD 1

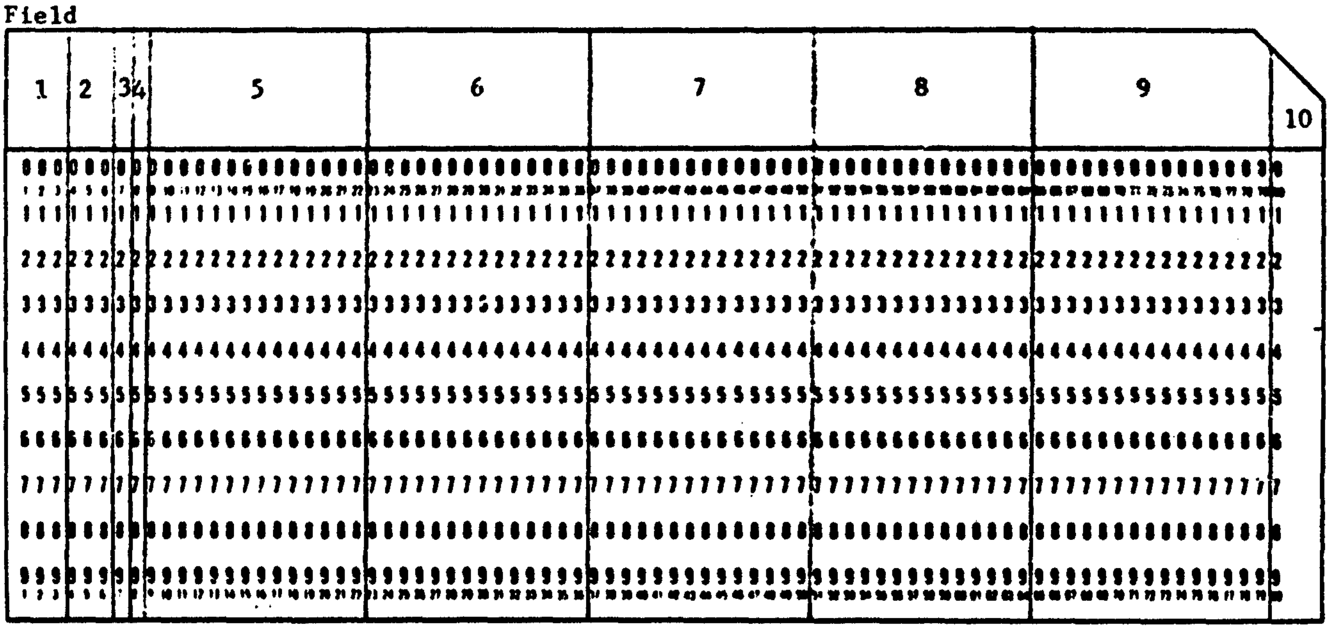
Field



Field No.	Columns	Description
1	1 - 3	Satellite Number - Justified right
2	4 - 6	Element set number
3	7	Not used
4	8	Card number (Card # = 2)
5	9 - 12	Year of T_o
6	13 - 22	Not used
7	23 - 36	T_o - Time of Epoch (day of year and fraction of day)
8	37 - 40	Not used
9	41 - 50	Not used
10	51 - 64	L_o - Mean Longitude - degrees
11	65 - 79	Not used
12	80	Card type E = Nodal Elements

FIGURE 4 ELEMENT CARD 2

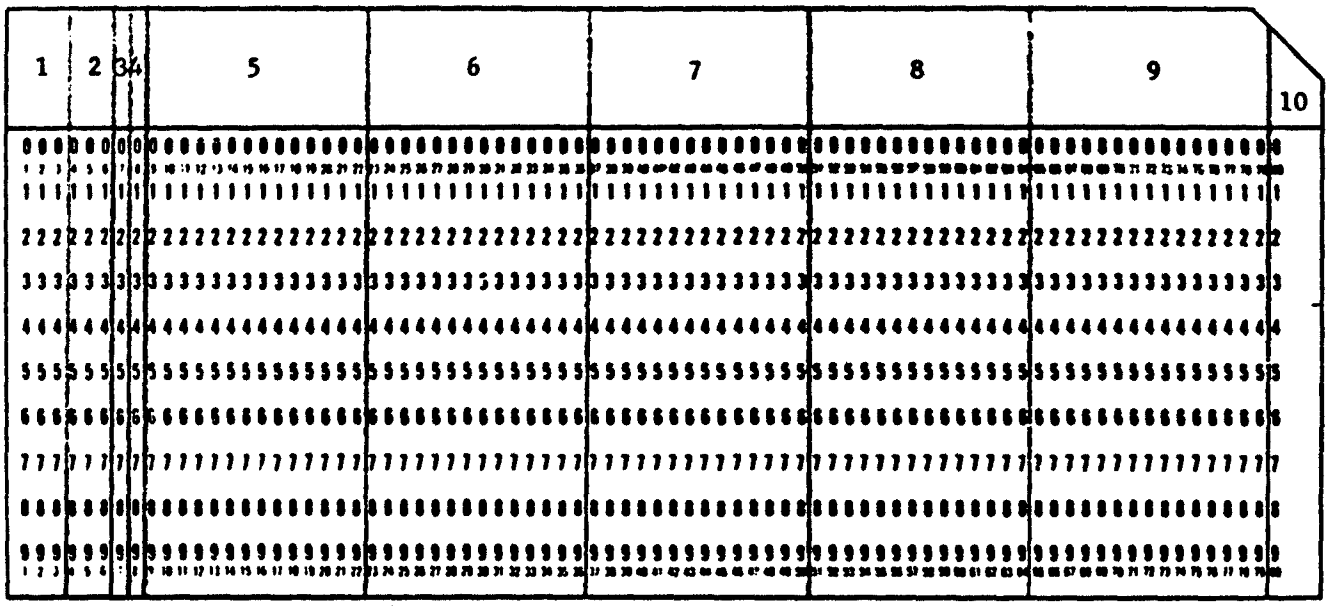
Field



Field No.	Columns	Description
1	1 - 3	Satellite number - justified right
2	4 - 6	Element set number - justified right
3	7	Not used
4	8	Card number (Card # = 3)
5	9 - 22	P_a - Anomalistic Period at Epoch - days/rev.
6	23 - 36	Ω_o - Right ascension of ascending node -degrees
7	3[illegible] - 50	ω_o - Argument of Perigee - degrees
8	51 - 64	q_o - Perigee distance - earth radii
9	65 - 79	Not used
10	80	Card type E - Nodal Elements

FIGURE 5 ELEMENT CARD 3

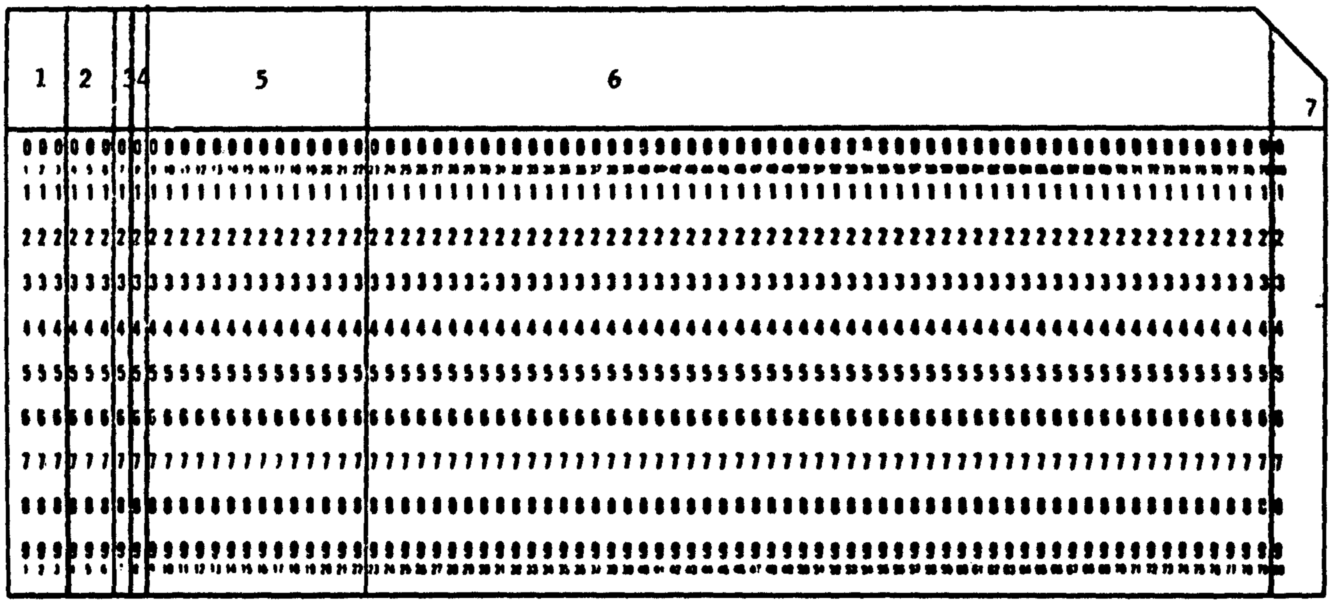
Field



Field No.	Columns	Description
1	1 - 3	Satellite number - justified right
2	4 - 6	Element set number - justified right
3	7	Not used
4	8	Card number (Card # = 4)
5	9 - 22	C - Rate of change of period - days/(rev)^2
6	23 - 36	$\dot{\Omega}_o$ - Time derivative of Right ascension of ascending node - degrees/day
7	37 - 50	$\dot{\omega}_o$ - Time derivative of argument of perigee (degree/day)
8	51 - 64	Not used
9	65 - 79	Not used
10	80	Card type E = Nodal Elements

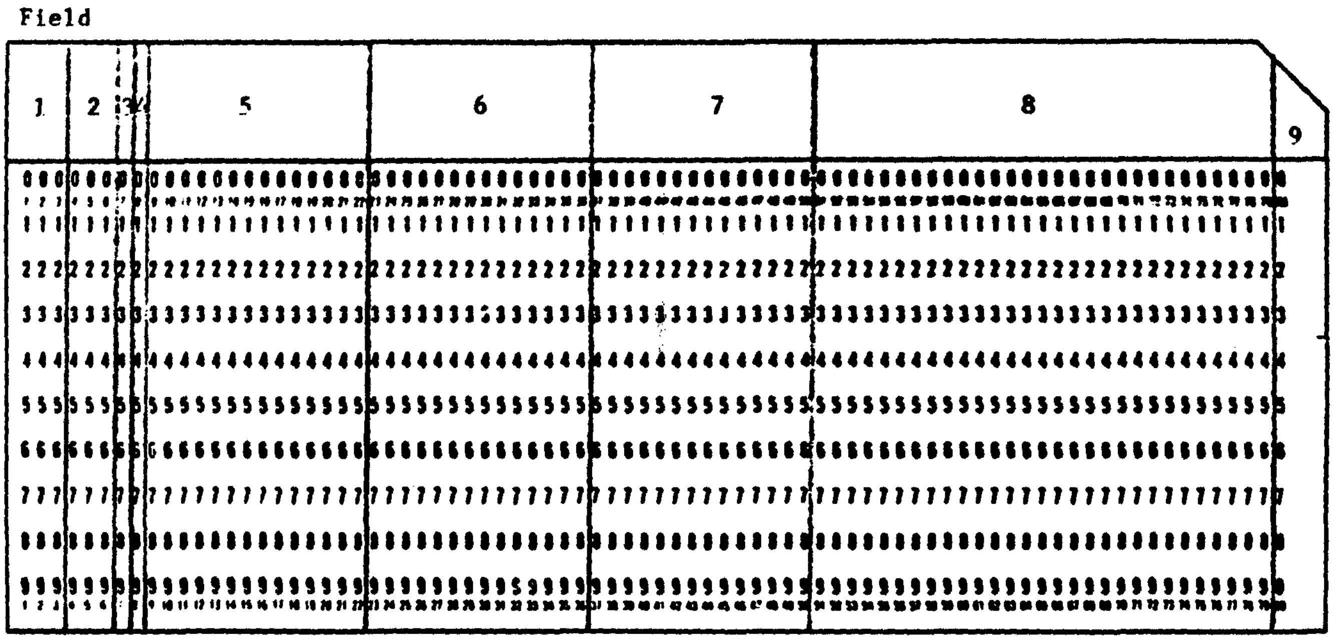
FIGURE 6 ELEMENT CARD 4

Field



Field No.	Columns	Description
1	1 - 3	Satellite number justified right
2	4 - 6	Element set number – justified right
3	7	Not used
4	8	Card number (Card # = 5)
5	9 - 22	d – decay acceleration
6	23 - 79	Not used
7	80	Card type E = Nodal Elements

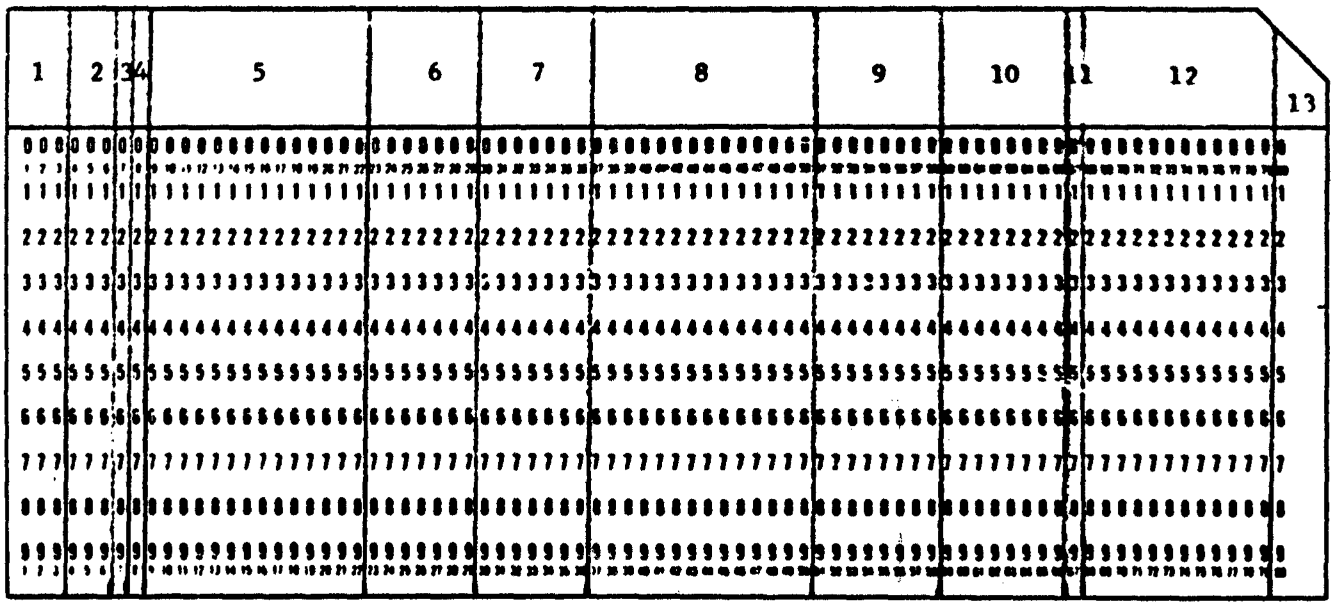
FIGURE 7 ELEMENT CARD 5



Field No.	Columns	Description
1	1 - 3	Satellite number - justified right
2	4 - 6	Element set number - justified right
3	7	Not used
4	8	Card number (Card # = 6)
5	9 - 22	a - semi-major axis - Earth radii
6	23 - 36	P_N - Nodal period - days/rev.
7	37 - 50	C_N - rate of change of nodal period - days/(rev)^2
8	51 - 79	Not used
9	80	Card type E = Nodal Elements

FIGURE 8 ELEMENT CARD 6

Field



Field No.	Columns	Description
1	1 - 3	Satellite number
2	4 - 6	Element set number
3	7	Not used
4	8	Card number (Card # = 7)
5	9 - 22	Not used
6	23 - 29	Initial Revolution, decimal may be punched in column 29
7	30 - 36	Final Revolution, decimal may be punched in column 36
8	37 - 50	Expiration date of Bulletin, in format; YMMDDHHMMSS.SS, decimal punched in column 48
9	51 - 58	RMS, in format XXXXX.XX; decimal punched in column 56
10	59 - 66	Number of observations used in obtaining RMS
11	67	ISTØP Blank or 0 = correct the inclination element 1 = do not correct the inclination 2 = do not correct the drag parameter 4 = correct time equation only
12	68 - 79	Not used
13	80	Card type E = Nodal Elements

FIGURE 9 ELEMENT CARD 7

APPENDIX C

DERIVATION OF PERTURBATIVE TERMS

For the actual derivation of the terms in Appendix B accounting for the asphericity of the Earth's potential, the reader is referred to the work of Kozai (1959)* and of Brouwer (1959). The intention here is to show that the SPADAT System formulas agree with those papers except in notation.

The cited papers supply some of the forms of the mathematical model, the constants of which are determined by fitting the observations. The remaining forms are designed to describe the effect of atmospheric drag on the satellite. The mean motion is described as a quadratic, and the mean longitude is represented by a cubic. The mean semi-major axis is related to the mean value of the mean motion by a modified form of Kepler's Harmonic Law. The perigee distance is assumed unaffected by drag.

The formulas of the Keplerian ellipse will not be derived here, even though their formulation in terms of the low-eccentricity parameters may be unfamiliar to the reader. Thus, this appendix will be restricted to those formulas containing the perturbative coupling factors: J_2, J_3, c" and d.

KEPLER'S HARMONIC LAW

Kepler's Harmonic Law is adopted in the form of Kozai's (1959) Equation 14, which in the notation of Appendix A is

$$n_o^2\, a_o^3 = k_e^2 \mu \left[1 - \frac{3}{2} J_2 \left(\frac{a_e}{p_o} \right)^2 \left(1 - \frac{3}{2} \sin^2 i_o\right) \sqrt{1 - e_o^2} \right]$$

*References are given at the end of this appendix.

It must be remembered that the subscripts refer to (mean) values at t_o. Kozai's a_0 (unperturbed semi-major axis) is not equivalent to a_o (mean at epoch), above. Equation (A-1) of Appendix A follows, if terms of order J_2^2 are omitted; Equation (B-2) is the exact inverse.

The translation of notation will be facilitated by reference to Brouwer's (1959) table on p. 396 and by comparing the potential given on that page, which, in SPADATS notation, is

$$U = \frac{k_e^2}{r}\left[1 - \sum_{j=2} J_j \left(\frac{a_e}{r}\right)^j P_j(\sin\delta)\right]$$

with Kozai's potential on p. 367:

$$U = \frac{GM}{r}\left[1 + \frac{A_2}{r^2}\left(\frac{1}{3} - \sin^2\delta\right) + \frac{A_3}{r^3}\left(\frac{5}{2}\sin^2\delta - \frac{3}{2}\right)\sin\delta + \ldots\right]$$

where $\sin\delta = \frac{z}{r}$

Thus,

$$A_3 = -J_3\, a_e^3$$

OTHER SECOND-HARMONIC TERMS

The nodal regression rate and movement of Perigee, Equations (A-5 and A-6) are derived by many authors. Kozai shows them on p. 372 as multipliers of elapsed time. Equation (A-9) is the sum of these rates, with the understanding that Ω is subtracted in retrograde orbits and that n_o is separated out of $\Delta\pi$ (See Equation (A-9) above).

THIRD-HARMONIC TERMS

The equivalent of Equation (A-7) is given by Kozai on p. 377. Here, Kozai's subscripted ω_o has the same meaning as herein, but the differences between e and e_o are periodic terms due to the geopotential harmonics, not drag as in Equation (A-7) above. Kozai states

"$e \cos\omega = e_o \cos\bar{\omega}$

$$e \sin\omega = e_o \sin\bar{\omega} + \frac{3}{4}\frac{A_3}{aA_2}\sin i$$

where $\bar{\omega} = \quad t + \omega_o$."

The perturbative term is

$$\frac{3}{4}\frac{A_3}{aA_2}\sin i = -\frac{1}{2}\frac{J_3}{J_2}\sin i\frac{a_e}{a}$$

This is equivalent to the term in Equation (A-7) only to order e^2. Combination of Brouwer's long-period terms in e and $g = \omega$ on p. 394 will also give Equation (A-7) and show that the effect on a_{xN} is of order e^2.

The third-harmonic term in the mean longitude, L_3, Equation (A-10) can be obtained by adding the three formulas of Brouwer, p. 391.

$$L_3 = \Delta_3 \ell' + \Delta_3 g' \pm \Delta_3 h'$$

where the lower sign corresponds to retrograde orbits. Thus in Brouwer's notation

$$L_3 = \frac{1}{4}\frac{\gamma_3}{\gamma_2}\cos g''\left(\frac{L'}{G''}\right)^2\left\{\frac{\sin I''}{e''}\left[1 - \left(\frac{G''}{L'}\right)^3\right] + \frac{e''}{\sin I''}\left[\frac{|H|}{G''} - \left(\frac{H}{G}\right)^2\right]\right\}$$

Translating this, but not combining, gives

$$L_3 = -\frac{1}{2}\frac{J_3}{J_2}\frac{a_e}{a_o}\cos\omega_o(1-e_o^2)^{-2}\left\{\frac{\sin i_o}{e_o}\left[1 - (1-e_o^2)^{3/2}\right] + \frac{e_o}{\sin i_o}\left[|\cos i_o| - \cos^2 i_o\right]\right\}$$

By expanding and omitting the e_o^2 terms, one obtains

$$L_3 = -\frac{1}{2}\frac{J_3}{J_2}\frac{a_e}{p_o}\cos\omega_o\left[\sin i_o\left(\frac{3}{2}e_o\right)\right.$$

$$\left.+\frac{e_o}{\sin i_o}\left(|\cos i_o| - \cos^2 i_o\right)\right]$$

This is transformed to eliminate the singularity at $i_o = 0$ or $i_o = 180^o$.

$$L_3 = -\frac{1}{2}\frac{J_3}{J_2}\frac{a_e}{p_o}e_o\cos\omega_o\sin i_o\left[\frac{\frac{3}{2}\sin^2 i_o + |\cos i_o| - \cos^2 i_o}{\sin^2 i_o}\right]$$

$$= -\frac{1}{2}\frac{J_3}{J_2}\frac{a_e}{p_o}a_{xN_o}\sin i_o\left[\frac{3+2|\cos i_o| - 5\cos^2 i_o}{2(1+|\cos i_o|)(1-|\cos i_o|)}\right]$$

(A-10) $$L_3 = -\frac{1}{4}\frac{J_3}{J_2}\frac{a_e}{p_o}a_{xN_o}\sin i_o\left[\frac{3+5|\cos i_o|}{1+|\cos i_o|}\right]$$

DRAG PERTURBATIONS

The form of the drag perturbations is mainly empirical. The determination of the observed rate of change of the orbit-period from the observational data appears at present to be a most efficient method of predicting future behavior of the satellite. A polynomial expansion of the time equation such as used for many years by SPACE TRACK (Findley 1962)

$$t_N = t_{N_o} + P_{N_o}(N-N_o) + c_o(N-N_o)^2 + d_o(N-N_o)^3$$

relates the nodal crossing time t_N to the number of revolutions $(N-N_o)$ with the coefficients: nodal period, rate of change of peiod and accelerative change in the period, P_N, c,d, respectively. The correction of the timing through the expression of the mean longitude has been more amenable to the SPADATS model. It is necessary, however, to establish the relationship between the foregoing equation and Equation (A-8).

In all that follows, the J_2 and J_3 terms will be omitted. The coupling of these terms with drag is negligible for most satellites; where it is not negligible, the present drag theory is inadequate, anyhow. This inadequacy is confined to the last few days of a satellite's life.

Derivatives may be taken of the old SPACE TRACK time equation:

$$1 = \frac{dN}{dt}\left[P_{N_o} + 2c_o(N-N_o) + 3d_o(N-N_o)^2\right]$$

Since

$$\frac{dN}{dt} = P_N^{-1}$$

$$P_N = P_{N_o} + 2c_o(N-N_o) + 3d_o(N-N_o)^2$$

Similarly,

$$\frac{dP_N}{dt} = \left[2c_o + 6d_o(N-N_o)\right]\frac{dN}{dt}$$

or, approximately,

$$2c_o = P_N \frac{dP_N}{dt}$$

From the mean longitude equation, (A-8),

(C-1) $$\frac{dL}{dt} = n = n_o\left[1 + 2c''(t-t_o) + 3d(t-t_o)^2\right]$$

(C-2) $$\frac{d^2L}{dt^2} = \frac{dn}{dt} = n_o\left[2c'' + 6d(t-t_o)\right]$$

(C-3) $$\frac{d^2n}{dt^2} = 6d\ n_o$$

The anomalistic period

(B-7) $$P_a = \frac{2\pi}{n_o}$$

is equal to the nodal period when J_2 is ignored. (See Equation B-10.) Thus,

$$2c_o \approx P_N \frac{dP_N}{dt} \approx P_a \frac{dP_a}{dt} = -P_a \frac{(2\pi)^2}{n^2} \frac{dn}{dt} \approx -\frac{(2\pi)^2}{n^3} 2c'' n_o$$

or

$$c_o \approx -\frac{(2\pi)^2}{n^2} c''$$

The factor of 1440 between the last equation and Equation (B-1) is due to the fact that the time is measured in days in the old SPACE TRACK equation and in minutes in the SPADATS model.

Equation (A-3) is designed to permit the cubic representation with the present 7-element correction. It is based on the work of Findley and must be accepted as purely empirical. The relevant experimentation is reported in Aeronutronic Monthly Status Report to 496L SPO, MPR-62-181, October, 1962, p. 9. Equation (A-3) may be replaced by an 8-element correction soon. The formulas of Appendix B are consistent with either source of d. The update formula (B-3) is, again, a result of Kepler's Harmonic Law in the form

(C-4) $$n = k_e \sqrt{\mu}\, a^{-3/2}$$

and Equation (C-1). Formula (B-4) results from Equation (C-2):

$$2n_o c''_p = \left(\frac{dn}{dt}\right)_{t_p} = n_o \left[2c'' + 6d(t_p - t_o)\right]$$

(B-4) $$c''_p = c'' + 3d(t_p - t_o)$$

At the old epoch (or the new epoch after the update), $t = t_o$

(C-5) $$\frac{dn}{dt} = 2c'' n_o$$

From Equation (C-4),

$$\frac{da}{dt} = -\frac{2}{3} \frac{a}{n} \frac{dn}{dt} = -\frac{2}{3} \frac{a^{5/2}}{k_e \sqrt{\mu}} \frac{dn}{dt}$$

The first form gives

(B-5) $$\left(\frac{da}{dt}\right)_o = -\frac{2}{3}\frac{a_o}{n_o}(2c'' n_o) = -\frac{4}{3}a_o c''$$

Differentiating the second form again, results in

$$\frac{d^2a}{dt^2} = -\frac{2}{3}\frac{a^{5/2}}{k_e\sqrt{\mu}}\frac{d^2n}{dt^2} - \frac{5}{3}\frac{a^{3/2}}{k_e\sqrt{\mu}}\frac{da}{dt}\frac{dn}{dt}$$

With the help of Equations (C-3), (C-4) and (C-5), this becomes, at t_o,

$$\left(\frac{d^2a}{dt^2}\right)_o = -\frac{2}{3}\frac{a_o}{n_o}(6d\, n_o) - \frac{5}{3n_o}\left(-\frac{4}{3}a_o c''\right)(2c'' n_o)$$

$$= -4\, a_o d + \frac{40}{9}\, a_o\, (c'')^2$$

which is Formula (B-6).

From the assumption that the perigee height does not change

$$\frac{dq}{dt} = \frac{d}{d}\left[a\,(1-e)\right] = 0$$

$$\frac{de}{dt} = \frac{1-e}{a}\frac{da}{dt}$$

With the aid of Equation (B-5), this becomes

(B-8) $$\left(\frac{de}{dt}\right)_o = \frac{1-e_o}{a_o}\left(-\frac{4}{3}a_o\, c''\right) = -\frac{4}{3}(1-e_o)\, c''$$

Finally, from (B-7)

$$\frac{dP_a}{dt} = - \frac{P_a}{n} \frac{dn}{dt}$$

(B-9) $$\left.\frac{dP_a}{dt}\right|_o = - 2c'' P_a$$

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